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On a density of the set of primes dividing the generalized Fibonacci numbers

By

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ABSTRACT J. C. Lagarias showed the set of prime numbers which divide some Lucas number L_n has positive density using Hasse's method [H]. In his paper he found several results for certain other special second-order linear recurrences [L], [W]. So we will research similar phenomena for slightly generalized second-order linear recurrences.

1 Introduction

In this note we will try to generalize a result of Lagalias on some second-order linear recurrences. Our method will be controlled by Hasse's one. Then we have to check whether these recurrences satisfy Hasse's conditions or not.

Now, any irreducible second-order recurrence $\{U_n\}$ whose terms U_n are rational numbers can be expressed in the form

$$U_n = \alpha\theta^n + \bar{\alpha}\bar{\theta}^n,$$

where α and θ are in the quadratic field K generated by the roots of the characteristic polinomial of $\{U_n\}$, and $\bar{\alpha}, \bar{\theta}$ are the algebraic conjugates of α, θ in K.

Hasse's conditions are as follows:

(1) $\theta/\bar{\theta} = \pm\phi^k$, where $k = 1$ or 2 for some ϕ in K,
(2) $\bar{\alpha}/\alpha = \zeta\phi^j$, where ζ is a root of unity in K and j is an integer.

We put $S_U = \{p : p \text{ is a prime and } p|U_n \text{ for some } n\}$. These particular recurrances $\{U_n\}$, which satifiy the above conditions (1) and (2), have a special property.

Definition 1 *A set Σ of primes is a Chebotarev set if and only if there is some finite normal extension L of the rationals $\boldsymbol{Q}$ such that a prime p is in Σ iff the Artin symbol* $\left[\dfrac{L/\boldsymbol{Q}}{(p)}\right]$ *is in specified conjugacy classes of the Galois group $Gal(L/\boldsymbol{Q})$.*

Definition 2 *Density $d(S_U)$ is defined*

$$\lim_{X\to\infty} \frac{\sharp S_{U,X}}{\sharp \mathbf{P}_X} = d(S_U),$$

where $\sharp S_{U,X} = \sharp\{p; p \in S_U\ p < X\}$ *and* $\sharp\mathbf{P}_X = \sharp\{p; p \text{ is a prime}, p < X\} \sim \dfrac{X}{\log X}$.

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Property 1 *Both the set S of primes and its complement*

$$\bar{S} = \{p : \ p \text{ is a prime and } \ p \notin S\}$$

have a decomposition into disjoint countable unions of Chebotarev sets of primes. That is

$$S = \bigcup_{j=1}^{\infty} S^{(j)}, \quad \bar{S} = \bigcup_{j=1}^{\infty} \bar{S}^{(j)},$$

where $S^{(j)}$ *and* $\bar{S}^{(j)}$ *are Chebotarev sets. Then the densities of the sets satisfy*

$$\sum_{j=1}^{\infty} d(S^{(j)}) + \sum_{j=1}^{\infty} d(\bar{S}^{(j)}) = 1.$$

If S is any set of primes having Property 1, then S has a natural density $d(S)$ given by

$$d(S) = \sum_{j=1}^{\infty} d(S^{(j)}).$$

2 Known results

Hasse and Lagarias obtained the following prime densities for several types of sequences:

Theorem 1 (H. Hasse [H]) *For the sequence* $\{V_n\} = \{2^n + 1\}$, *the set of primes*

$$\begin{aligned} S_V \ &= \{p : \ p \text{ is a prime and } p \text{ divies } 2^n + 1 \text{ for some } n \geq 0\} \\ &= \{p \in \mathbf{P}; \ p | V_n \text{for some } n\}. \end{aligned}$$

has density $d(S_V) = \dfrac{17}{24}$.

Hasse's result actually covers all the sequences

$$\{A_n\} = \{a^n + 1 \mid n \geq 0\},$$

where a is an integer ≥ 3, and the density of the associated set $S_A = \{p \in \mathbf{P} : p | A_n \text{for some } n\}$ is

$$d(S_A) = \frac{2}{3}.$$

Theorem 2 (J. C. Lagarias [L]) *For the sequence* $\{L_n\}$ ($L_{n+1} = L_n + L_{n-1}$, $L_1 = 2$, $L_2 = 1$), *the set of primes*

$$S_L = \{p \in \mathbf{P} \ ; \ p | L_n \ \text{for some } n\}$$

has density $d(S_L) = \dfrac{2}{3}$.

Theorem 3 (J. C. Lagarias [L2]) *For the sequence* $\{W_n\}$ $(W_n = 5W_{n-1} - 7W_{n-2}, W_0 = 1, W_1 = 2)$, *the set of primes*

$$S_W = \{p \in \mathbf{P} \ ; \ p|W_n \ for \ some \ n\}$$

has density $d(S_W) = \dfrac{3}{4}$.

Lagarias considered

$$\{A_n(m)\}, \quad \{B_n(m)\} \quad (m : \text{fixed})$$

where both series admit the condotion:

$$U_n = mU_{n-1} - U_{n-2}$$

with $A_0(m) = B_0(m) = 1, A_1(m) = m + 1, B_1(m) = m - 1$, to which Hasse's method is applicable. In the cases of fields $K = \boldsymbol{Q}\left(\sqrt{m^2 - 4}\right)$, for the following sets of primes:

$$\begin{aligned} S_A(m) = & \ \{p \in \mathbf{P};\ p|A_n(m) \ for \ some \ n\}, \\ S_B(m) = & \ \{p \in \mathbf{P};\ p|B_n(m) \ for \ some \ n\}, \end{aligned}$$

it is known that $d(S_A(m)) = d(S_B(m)) = \dfrac{1}{3}$.

3 Theorem

Let

$$\{U_n\} \ (U_n = mU_{n-1} + U_{n-2}, \ U_0 = 2, \ U_1 = m),$$

be a second-order linear recurrence, where we assume that $D = m^2 + 4$ is a prime discriminant of $K = \boldsymbol{Q}\left(\sqrt{D}\right)$. Then we have
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Theorem 4 *For the sequence* $\{U_n\}$ $(U_n = mU_{n-1} + U_{n-2}, \ U_0 = 2, \ U_1 = m)$, *the set of primes*

$$S_U = \{p \in \mathbf{P} \ ; \ p|U_n \ for \ some \ n\}$$

has density $d(S_U) = \dfrac{2}{3}$.

Remark 1 In the case of $m = 1$, the theorem above coincides with Theorem 2. We can prove Theorem 4 by a similar way to Theorem 2.

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